J. Math. Anal. Appl. 550 (2025) 129442

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Analytical properties of derivative polynomials

Guo-Niu Han [a], Hailing Li [b], Shi-Mei Ma [b,*], Jean Yeh [c,*], Yeong-Nan Yeh [d]

[a] *I.R.M.A., UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, F-67084 Strasbourg, France*
[b] *School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, Shandong, China*
[c] *Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan*
[d] *College of Mathematics and Physics, Wenzhou University, Wenzhou 325035, China*

A R T I C L E   I N F O

Article history:
Received 23 September 2024
Available online 6 March 2025
Submitted by M.J. Schlosser

Keywords:
Chebyshev polynomials
Derivative polynomials
Tangent
Secant

A B S T R A C T

We study analytical properties of derivative polynomials for tangent and secant, including recurrence relations, explicit formulas and expansion formulas. Firstly, we discuss the connections between central binomial coefficients and trigonometric functions. Secondly, we explore the similarity of derivative polynomials and Chebyshev polynomials. The idea is to choose the derivative polynomials as basis sets of a polynomial space. From this viewpoint, we give an expansion of the derivative polynomials for tangent in terms of the derivative polynomials for secant as well as a result in the reverse direction. Moreover, we get the Frobenius-type formulas for exterior peak and left peak polynomials. Finally, we discuss the connections between derivative polynomials and Eulerian polynomials.

1. Introduction

Hoffman [17] once said in his paper on derivative polynomials: “Sometimes problems naturally occur in pairs, and it is best to tackle both at the same time.” This idea was fleshed out in the paper of Hetyei [15]. The derivative polynomials for tangent and secant obey this principle. Another classical pair of polynomials that naturally occurs in pairs is the pair of Chebyshev polynomials of the first and second kinds. All polynomial sequences considered in this paper form analogous pairs. This paper is motivated by exploring the similarity of derivative polynomials and Chebyshev polynomials.

An elementary result in the theory of trigonometry says that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta = 1+\tan^2\theta, \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta = \tan\theta\sec\theta. \end{cases} \tag{1}$$

* Corresponding authors.
E-mail addresses: guoniu.han@unistra.fr (G.-N. Han), hailing20042006@163.com (H. Li), shimeimapapers@163.com (S.-M. Ma), chunchenyeh@mail.nknu.edu.tw (J. Yeh), mayeh@math.sinica.edu.tw (Y.-N. Yeh).

https://doi.org/10.1016/j.jmaa.2025.129442

The *derivative polynomials* for tangent and secant are respectively defined as follows:

$$\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\tan\theta = P_n(\tan\theta),\ \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\sec\theta = \sec\theta\, Q_n(\tan\theta).$$

By Taylor's theorem, we have

$$\tan(\theta+z) = \sum_{n=0}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\tan(\theta)\frac{z^n}{n!} = \sum_{n=0}^{\infty}P_n(\tan\theta)\frac{z^n}{n!},$$
$$\sec(\theta+z) = \sum_{n=0}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\sec(\theta)\frac{z^n}{n!} = \sec(\theta)\sum_{n=0}^{\infty}Q_n(\tan\theta)\frac{z^n}{n!}.$$

The tangent formula says that

$$\tan(\theta+z) = \frac{\tan(\theta)+\tan(z)}{1-\tan(\theta)\tan(z)}.$$

Since $\sec(\theta+z) = \frac{1}{\cos(\theta+z)}$, it follows from cosine formula that

$$\sec(\theta+z) = \frac{1}{\cos(\theta)\cos(z)-\sin(\theta)\sin(z)} = \frac{\sec(\theta)\sec(z)}{1-\tan(\theta)\tan(z)}.$$

So we obtain

$$P(x;z) = \sum_{n=0}^{\infty}P_n(x)\frac{z^n}{n!} = \frac{x+\tan z}{1-x\tan z},\ Q(x;z) = \sum_{n=0}^{\infty}Q_n(x)\frac{z^n}{n!} = \frac{\sec z}{1-x\tan z}. \tag{2}$$

It should be noted that Carlitz and Scoville [8] deduced (2) by using the method of characteristics.

The study of derivative polynomials was initiated by Knuth and Buckholtz [21]. Using the chain rule, they deduced that

$$P_{n+1}(x) = (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x),\ Q_{n+1}(x) = (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}Q_n(x) + xQ_n(x). \tag{3}$$

Below are these polynomials for $n \leqslant 3$:

$$P_0(x) = x,\ P_1(x) = 1+x^2,\ P_2(x) = 2x+2x^3,\ P_3(x) = 2+8x^2+6x^4,$$
$$Q_0(x) = 1,\ Q_1(x) = x,\ Q_2(x) = 1+2x^2,\ Q_3(x) = 5x+6x^3.$$

Note that $P_n(-x) = (-1)^{n+1}P_n(x)$ and $Q_n(x) = (-1)^nQ_n(-x)$. Hence $P_n(x)$ and $Q_n(x)$ are both alternately even and odd.

The derivative polynomials can be used to express some improper integrals and infinite series, including Hurwitz zeta functions and Dirichlet L-series, see [1,6,12,17,18,28]. For $x > 0$, the *gamma function* $\Gamma(x)$ and the *digamma function* $\psi(x)$ are defined by

$$\Gamma(x) = \int_0^{\infty}\mathrm{e}^{-t}t^{x-1}dt,\ \psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Let $\psi_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\psi(x)$ be the *polygamma functions* for $n \geqslant 1$. It is well known that

$$\psi_n(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, x),$$

where $\zeta(n, x)$ is the Hurwitz zeta function. Polygamma functions arise naturally in the study of beta distributions, and they obey the reflection formula (see [2,4]):

$$\psi_n(1-x) + (-1)^{n+1} \psi_n(x) = (-1)^n \pi \frac{\mathrm{d}^n}{\mathrm{d}x^n} \cot(\pi x) = \pi^{n+1} P_n(\cot(\pi x)),$$

where the last equality follows from the fact that $\frac{\mathrm{d}^n}{\mathrm{d}x^n} \cot(x) = (-1)^n P_n(\cot(x))$.

In the next section we collect the definitions, notation and preliminary results that will be used in the rest of this work. In Section 3, we present the connections between central binomial coefficients and trigonometric functions. In Section 4, we explore the similarity of derivative polynomials and Chebyshev polynomials. In particular, we get the Frobenius-type formulas for exterior peak and left peak polynomials. In Section 5, we establish the connection between derivative polynomials and Eulerian polynomials.

2. Preliminaries

2.1. Chebyshev polynomials

The *Chebyshev polynomials of the first kind* are defined by

$$T_n(x) = \cos(n\theta), \text{ when } x = \cos(\theta).$$

They are orthogonal on $[-1, 1]$ with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$, see [9] for instance. While the *Chebyshev polynomials of the second kind* are defined by

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \text{ when } x = \cos(\theta).$$

The polynomials $U_n(x)$ are orthogonal on $[-1, 1]$ with respect to $\sqrt{1-x^2}$. Explicitly, we have

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k}, \; U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} (x^2-1)^k x^{n-2k}, \tag{4}$$

which imply that $T_n(x)$ and $U_n(x)$ are both alternately even and odd, see [26] for details. The identity $\sin((n+1)\theta) - \sin((n-1)\theta) = 2\sin(\theta)\cos(n\theta)$ leads to a well known relationship:

$$T_n(x) = \frac{1}{2} \left(U_n(x) - U_{n-2}(x) \right). \tag{5}$$

2.2. Explicit formulas of derivative polynomials

In [1], Adamchik solved a long-standing problem of finding a closed-form expression for the higher derivatives of the cotangent function:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \cot(x) = (2\imath)^n (\cot(x) - \imath) \sum_{k=1}^{n} \frac{k!}{2^k} \left\{ {n \atop k} \right\} (\imath \cot(x) - 1)^k,$$

where $\imath = \sqrt{-1}$ and $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind, i.e., the number of ways of partitioning the set $[n] := \{1, 2, \ldots, n\}$ into k blocks. Equivalently, we have

$$P_n(x) = (-2\imath)^n (x-\imath) \sum_{k=1}^{n} \frac{k!}{2^k} \left\{ {n \atop k} \right\} (\imath x - 1)^k.$$

Since then, there has been much progress in the coefficients of the derivative polynomials, see [6,13,20,23,28]. For example, in [6], Boyadzhiev obtained that

$$Q_n(x) = \imath^n \sum_{j=0}^{n} \left[(-1)^j j! \sum_{k=j}^{n} \binom{n}{k} \left\{ {k \atop j} \right\} 2^{k-j} \right] (x+1)^j.$$

2.3. Permutation statistics, Eulerian polynomials, Euler numbers and Springer numbers

Let $\mathfrak{S}_n$ denote the set of all permutations of $[n] := \{1, 2, 3, \dots, n\}$. For any $\pi \in \mathfrak{S}_n$, written as the word $\pi(1)\pi(2)\cdots\pi(n)$, the entry $\pi(i)$ is called

- a *descent* if $\in [n-1]$ and $\pi(i) > \pi(i+1)$;
- a *double descent* if $i \in \{2, 3, \dots, n\}$ and $\pi(i-1) > \pi(i) > \pi(i+1)$, where we set $\pi(n+1) = 0$;
- an *interior peak* if $i \in \{2, 3, \dots, n-1\}$ and $\pi(i-1) < \pi(i) > \pi(i+1)$;
- a *left peak* if $i \in [n-1]$ and $\pi(i-1) < \pi(i) > \pi(i+1)$, where we set $\pi(0) = 0$;
- an *exterior peak* if $i \in [n]$ and $\pi(i-1) < \pi(i) > \pi(i+1)$, where we set $\pi(0) = \pi(n+1) = 0$.

Let $\mathrm{des}\,(\pi)$ (resp. $\mathrm{ddes}\,(\pi)$, $\mathrm{ipk}\,(\pi)$, $\mathrm{lpk}\,(\pi)$, $\mathrm{epk}\,$) denote the number of descents (resp. double descents, interior peaks, left peaks, exterior peaks) of π. For example, if $\pi = 214356$, then $\mathrm{des}\,(\pi) = 2$, $\mathrm{ddes}\,(\pi) = 0$, $\mathrm{ipk}\,(\pi) = 1$, $\mathrm{lpk}\,(\pi) = 2$ and $\mathrm{epk}\, = 3$.

The Eulerian polynomials $A_n(x)$ first introduced by Leonhard Euler in the series summations:

$$\sum_{k=0}^{\infty} k^n x^k = \frac{xA_n(x)}{(1-x)^{n+1}}.$$

A combinatorial interpretation of $A_n(x)$ is given as follows (see [19,25]):

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}\,(\pi)}.$$

The Eulerian polynomial admits several remarkable expansions in terms of different polynomial bases. Here is the classical *Frobenius formula* for Eulerian polynomials (see [11]):

$$xA_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k (1-x)^{n-k}. \tag{6}$$

In Corollary 9, we provide similar formulas for exterior peak and left peak polynomials.

We say that $\pi \in \mathfrak{S}_n$ is *alternating* if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$, i.e., $\pi(2i-1) > \pi(2i)$ and $\pi(2i) < \pi(2i+1)$ for $1 \leqslant i \leqslant \lfloor n/2 \rfloor$. A famous result of André [3] says that

$$\sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \tan z + \sec z = 1 + z + \frac{z^2}{2!} + 2\frac{z^3}{3!} + 5\frac{z^4}{4!} + 16\frac{z^5}{5!} + \cdots, \tag{7}$$

where E_n is the number of alternating permutations in $\mathfrak{S}_n$. Since Euler used (7) as the definition of E_n, the numbers E_n are called *Euler numbers* (sometimes they are called *André numbers*).

Note that

$$\sum_{n=0}^{\infty} E_{2n+1}\frac{z^{2n+1}}{(2n+1)!} = \tan z, \quad \sum_{n=0}^{\infty} E_{2n}\frac{z^{2n}}{(2n)!} = \sec z.$$

For this reason, the numbers E_{2n+1} are sometimes called *tangent numbers* and E_{2n} are called *secant numbers*. The reader is referred to [31] for a survey on this subject. In [21], Knuth-Buckholtz noted that

$$P_{2n+1}(0) = E_{2n+1}, \; Q_{2n}(0) = E_{2n}.$$

Hoffman [17] found that

$$P_n(1) = 2^n(P_n(0) + Q_n(0)) = \begin{cases} 2^n Q_n(0) = 2^{2k}E_{2k}, & \text{if } n = 2k \text{ is even;} \\ 2^n P_n(0) = 2^{2k+1}E_{2k+1}, & \text{if } n = 2k+1 \text{ is odd.} \end{cases} \tag{8}$$

He also noted that $Q_n(1)$ are the Springer numbers of root systems of type B_n, see [18, Proposition 4.1]. A *snake* of type B_n is a sequence $(x_1, x_2, \ldots, x_n)$ of integers such that $0 < x_1 > x_2 < \cdots x_n$ and $\{|x_1|, |x_2|, \ldots, |x_n|\} = [n]$, i.e., $|x_1||x_2|\cdots|x_n|$ is an alternating permutation in $\mathfrak{S}_n$. Let s_n be the number of snakes of type B_n. Following [18, Theorem 4.2], one has $s_n = Q_n(1)$, and so

$$\sum_{n=0}^{\infty} s_n \frac{z^n}{n!} = \frac{1}{\cos z - \sin z}.$$

A main result obtained by Hoffman [18, Theorem 3.1] says that

$$Q_n(1) = -\sin\frac{n\pi}{2} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}(-1)^k P_{n-2k}(1), \tag{9}$$

which is closely related to the computation of the $Q_n(1)$ from the numbers $P_n(1)$ via Seidel matrices. In [15], Hetyei showed that derivative polynomials are closely related to the face enumerating polynomials of the Chebyshev transforms of the Boolean algebras. He also showed that the zeros of $P_n(x)$ and $Q_n(x)$ are pure imaginary, have multiplicity 1, belong to the line segment $[-\imath, \imath]$, where $\imath = \sqrt{-1}$, see [15, Corollary 8.7].

2.4. Variants of (1)

Setting $s(\theta) = \sec\theta$ and $t(\theta) = \tan\theta$, since $1 + \tan^2\theta = \sec^2\theta$, an equivalent variant of (1) is given as follows:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\theta}s(\theta) = s(\theta)t(\theta), \\ \frac{\mathrm{d}}{\mathrm{d}\theta}t(\theta) = s^2(\theta), \end{cases} \tag{10}$$

which can be used to study the interior and left peak polynomials (see [10,22,27] for details). If we define $f = \sec(\sqrt{q}\theta)$ and $g = \sqrt{q}\tan(\sqrt{q}\theta)$, where $q > 0$ is a constant, then $\frac{\mathrm{d}}{\mathrm{d}\theta}f = fg$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}g = qf^2$, which yield the differential system:

$$\begin{cases} f \to fg, \\ g \to qf^2. \end{cases} \tag{11}$$

For the differential system (10), Ma [22, Theorem 1] found that

$$\begin{cases} \frac{\mathrm{d}^n}{\mathrm{d}\theta^n} s(\theta) = \sum_{\pi\in\mathfrak{S}_n} s(\theta)^{2\mathrm{lpk}\,(\pi)+1} t(\theta)^{n-2\mathrm{lpk}\,(\pi)}, \\ \frac{\mathrm{d}^n}{\mathrm{d}\theta^n} t(\theta) = \sum_{\pi\in\mathfrak{S}_n} s(\theta)^{2\mathrm{ipk}\,(\pi)+2} t(\theta)^{n-1-2\mathrm{ipk}\,(\pi)}. \end{cases}$$

Note that $\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{epk}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{ipk}\,(\pi)+1}$. Chen and Fu [10, Theorem 9] observed that

$$\frac{\mathrm{d}^n}{\mathrm{d}\theta^n} t(\theta) = \sum_{\pi\in\mathfrak{S}_n} s(\theta)^{2\mathrm{epk}\,(\pi)} t(\theta)^{n+1-2\mathrm{epk}\,(\pi)}.$$

We shall end this section by giving an application of (2). Let us put

$$a = P(x;z) = \frac{x+\tan z}{1-x\tan z},\ b = Q(x;z) = \frac{1}{\cos z - x\sin z}.$$

Consider the differentiations of a and b with respect to z while x is being fixed. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} a = \frac{1+x^2}{(\cos z - x\sin z)^2},\ \frac{\mathrm{d}}{\mathrm{d}z} b = \frac{x\cos z + \sin z}{(\cos z - x\sin z)^2}.$$

So we get the following differential system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}z} a = (1+x^2)b^2, \\ \frac{\mathrm{d}}{\mathrm{d}z} b = ab, \end{cases}$$

which gives a variant of (11).

3. Trigonometric functions and central binomial coefficients

In 1972, Beeler et al. [5] found an elegant identity:

$$\tan(n\arctan(t)) = \frac{1}{\imath}\frac{(1+\imath t)^n - (1-\imath t)^n}{(1+\imath t)^n + (1-\imath t)^n},$$

which can be simplified to

$$\tan(nx) = \frac{\sum_{k\geqslant 0}(-1)^k \binom{n}{2k+1}\tan^{2k+1}(x)}{\sum_{k\geqslant 0}(-1)^k \binom{n}{2k}\tan^{2k}(x)}, \tag{12}$$

where $x = \arctan(t)$ and $\imath = \sqrt{-1}$. It is natural to further explore the connections between central binomial coefficients and trigonometric functions.

Consider the following formal computations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)^{n+1} = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)^{n} \frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta),\ \left(\sec(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{n+1} = \left(\sec(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{n}\sec(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}.$$

As a dual of (4), we can now present the following result.

Theorem 1. *For $n \geqslant 1$, we have*

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)^n (\sec(\theta)) = n!\sum_{k\geqslant 0}\binom{n+1}{2k+1}\tan^{n-2k}(\theta)\sec^{n+2k+1}(\theta), \\ \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)^n (\tan(\theta)) = n!\sum_{k\geqslant 0}\binom{n+1}{2k}\tan^{n-2k+1}(\theta)\sec^{n+2k}(\theta). \end{cases}$$

In other words, we have

$$\begin{cases} \left(\sec(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^n (\sec^2(\theta)) = n!\sum_{k\geqslant 0}\binom{n+1}{2k+1}\tan^{n-2k}(\theta)\sec^{n+2k+2}(\theta), \\ \left(\sec(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^n (\tan(\theta)\sec(\theta)) = n!\sum_{k\geqslant 0}\binom{n+1}{2k}\tan^{n-2k+1}(\theta)\sec^{n+2k+1}(\theta). \end{cases}$$

Proof. Set

$$M(n,k) = n!\binom{n+1}{2k+1},\ N(n,k) = n!\binom{n+1}{2k}.$$

Using the recurrence relations

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},\ \binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

and it can be easily verified that

$$M(n+1,k) = (n+2k+2)M(n,k) + (n-2k+2)M(n,k-1),$$
$$N(n+1,k) = (n+2k+1)N(n,k) + (n-2k+3)N(n,k-1).$$

Note that

$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)\sec(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\sec^2(\theta) = 2\tan(\theta)\sec^2(\theta),$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)\tan(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\tan(\theta) = \tan^2(\theta)\sec(\theta) + \sec^3(\theta).$$

So the expansions hold for $n = 1$. Assume that there exist nonnegative integers $\widetilde{M}(n,k)$ and $\widetilde{N}(n,k)$ such that

$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)^n (\sec(\theta)) = \sum_{k\geqslant 0}\widetilde{M}(n,k)\tan^{n-2k}(\theta)\sec^{n+2k+1}(\theta),$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)^n (\tan(\theta)) = \sum_{k\geqslant 0}\widetilde{N}(n,k)\tan^{n-2k+1}(\theta)\sec^{n+2k}(\theta).$$

Then we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sec(\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta)\right)^n (\sec(\theta))\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sum_{k\geqslant 0}\widetilde{M}(n,k)\tan^{n-2k}(\theta)\sec^{n+2k+2}(\theta)\right) \\ &= \sum_{k\geqslant 0}(n-2k)\widetilde{M}(n,k)\tan^{n-2k-1}(\theta)\sec^{n+2k+4}(\theta) + \\ &\sum_{k\geqslant 0}(n+2k+2)\widetilde{M}(n,k)\tan^{n-2k+1}(\theta)\sec^{n+2k+2}(\theta), \end{aligned}$$

so we get $\widetilde{M}(n+1,k) = (n+2k+2)\widetilde{M}(n,k) + (n-2k+2)\widetilde{M}(n,k-1)$. The numbers $M(n,k)$ and $\widetilde{M}(n,k)$ satisfy the same recurrence relation and initial conditions, so they agree. Similarly, it is routine to verify that $N(n,k) = \widetilde{N}(n,k)$. This completes the proof. □

Corollary 2. *For $n \geqslant 1$, we have*

$$\left(\sec(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{n-1}\frac{\mathrm{d}}{\mathrm{d}\theta}(\tan(\theta)+\sec(\theta)) = (n-1)!\sec^n(\theta)(\tan(\theta)+\sec(\theta))^n.$$

4. The similarity of derivative polynomials and Chebyshev polynomials

4.1. Derivative polynomial bases

Motivated by (5), we shall express $Q_{n+1}(x)$ in terms of $\{P_i(x)\}_{i=-1}^{n}$, where we set $P_{-1}(x) = 1$, since $P_{-1}(x)$, $P_0(x)$, $P_1(x)$, $P_2(x)$, $\ldots$, $P_n(x)$ form a basis for polynomials with degree less than or equal to $n+1$. We can now present the first result of this section.

Theorem 3. *We have*

$$Q_{2n}(x) = (-1)^n + \sum_{k=0}^{n-1}\binom{2n}{2k+1}(-1)^k P_{2n-2k-1}(x), \tag{13}$$

$$Q_{2n+1}(x) = \sum_{k=0}^{n}\binom{2n+1}{2k+1}(-1)^k P_{2n-2k}(x). \tag{14}$$

Proof. By using (2), we obtain

$$\begin{aligned}
Q(x;z) &= \frac{\sin^2(z)+\cos^2(z)}{\cos(z)}\frac{1}{1-x\tan(z)} \\
&= \left(\cos(z) - x\sin(z) + x\sin(z) + \frac{\sin^2(z)}{\cos(z)}\right)\frac{1}{1-x\tan(z)} \\
&= \frac{\cos(z)-x\sin(z)}{1-x\tan(z)} + \sin(z)\frac{x+\tan(z)}{1-x\tan(z)}.
\end{aligned}$$

After simplifying, we get

$$Q(x;z) = \cos(z) + \sin(z)P(x;z). \tag{15}$$

So we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_{2n}(x)\frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} Q_{2n+1}(x)\frac{z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!} + \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!} \times \left(\sum_{n=1}^{\infty}\frac{P_{2n-1}(x)}{(2n-1)!}z^{2n-1} + \sum_{n=0}^{\infty}\frac{P_{2n}(x)}{(2n)!}z^{2n}\right)
\end{aligned}$$

Selecting the coefficients of z^{2n} and z^{2n+1}, we arrive at

$$\frac{Q_{2n}(x)}{(2n)!} = \frac{(-1)^n}{(2n)!} + \sum_{k=0}^{n-1}\frac{(-1)^k}{(2k+1)!}\frac{P_{2n-2k-1}(x)}{(2n-2k-1)!},$$

$$\frac{Q_{2n+1}(x)}{(2n+1)!} = \sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}\frac{P_{2n-2k}(x)}{(2n-2k)!},$$

and so the proof is complete. □

Example 4. In the case when $n = 2$ in Theorem 3, we have

$$Q_4(x) = 5 + 28x^2 + 24x^4 = 4P_3(x) - 4P_1(x) + 1,$$
$$Q_5(x) = 61x + 180x^3 + 120x^5 = 5P_4(x) - 10P_2(x) + P_0(x).$$

It follows from Theorem 3 that

$$Q_{2n}(1) = (-1)^n + \sum_{k=0}^{n-1} \binom{2n}{2k+1} (-1)^k P_{2n-2k-1}(1),$$
$$Q_{2n+1}(1) = \sum_{k=0}^{n} \binom{2n+1}{2k+1} (-1)^k P_{2n-2k}(1),$$

which differ from (9). Combining the above formulas with (8), we obtain the following result.

Corollary 5. *For any $n \geqslant 0$, we have*

$$s_{2n} = (-1)^n + \sum_{k=0}^{n-1} \binom{2n}{2k+1} (-1)^k 2^{2n-2k-1} E_{2n-2k-1},$$
$$s_{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k+1} (-1)^k 2^{2n-2k} E_{2n-2k}.$$

The *Bernoulli numbers* B_n can be defined by the exponential generating function

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \frac{1}{6}\frac{z^2}{2!} - \frac{1}{30}\frac{z^4}{4!} + \frac{1}{42}\frac{z^6}{6!} - \frac{1}{30}\frac{z^8}{8!} + \cdots .$$

In particular, $B_{2n+1} = 0$ for $n \geqslant 1$, since $\frac{z}{2}\coth\left(\frac{z}{2}\right)$ is an even function and

$$\frac{z}{2}\coth\left(\frac{z}{2}\right) = \frac{z}{2}\frac{\mathrm{e}^{z/2} + \mathrm{e}^{-z/2}}{\mathrm{e}^{z/2} - \mathrm{e}^{-z/2}} = \frac{z}{\mathrm{e}^z - 1} + \frac{z}{2}.$$

The Bernoulli numbers appear often in the coefficients of trigonometric functions (see [14, Chapter 6] for details). As illustrations, we have

$$z\csc(z) = 1 + \sum_{n=1}^{\infty} (-1)^{n+1}(4^n - 2) B_{2n} \frac{z^{2n}}{(2n)!},$$
$$z\cot(z) = \sum_{n=0}^{\infty} (-4)^n B_{2n} \frac{z^{2n}}{(2n)!} \qquad (0 < |z| < \pi).$$

It follows from (15) that $P(x; z) = \csc(z)Q(x; z) - \cot(z)$. Thus

$$zP(x; z) = z\csc(z)Q(x; z) - z\cot(z).$$

So we have

$$\sum_{n=0}^{\infty} P_{2n}(x)\frac{z^{2n+1}}{(2n)!} + \sum_{n=0}^{\infty} P_{2n+1}(x)\frac{z^{2n+2}}{(2n+1)!}$$
$$= \left(1+\sum_{i=0}^{\infty}(-1)^{i+2}(4^{i+1}-2)B_{2i+2}\frac{z^{2i+2}}{(2i+2)!}\right)\left(\sum_{n=0}^{\infty} Q_{2n}(x)\frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} Q_{2n+1}(x)\frac{z^{2n+1}}{(2n+1)!}\right) -$$
$$\sum_{n=0}^{\infty}(-4)^n B_{2n}\frac{z^{2n}}{(2n)!}.$$

Equating the coefficients of z^{2n+1} and z^{2n+2}, we see that

$$\frac{P_{2n}(x)}{(2n)!} = \frac{Q_{2n+1}(x)}{(2n+1)!} + \sum_{i=0}^{n-1} \frac{(-1)^{i+2}(4^{i+1}-2)B_{2i+2}}{(2i+2)!}\frac{Q_{2n-2i-1}(x)}{(2n-2i-1)!},$$
$$\frac{P_{2n+1}(x)}{(2n+1)!} = \frac{Q_{2n+2}(x)}{(2n+2)!} + \sum_{i=0}^{n} \frac{(-1)^{i+2}(4^{i+1}-2)B_{2i+2}}{(2i+2)!}\frac{Q_{2n-2i}(x)}{(2n-2i)!} - \frac{(-4)^{n+1}B_{2n+2}}{(2n+2)!}.$$

Using these expressions, we get the following result.

Theorem 6. *We have*

$$(2n+1)P_{2n}(x) = Q_{2n+1}(x) + \sum_{i=0}^{n-1}\binom{2n+1}{2i+2}(-1)^{i+2}(4^{i+1}-2)B_{2i+2}Q_{2n-2i-1}(x),$$
$$(2n+2)P_{2n+1}(x) = Q_{2n+2}(x) + \sum_{i=0}^{n}\binom{2n+2}{2i+2}(-1)^{i+2}(4^{i+1}-2)B_{2i+2}Q_{2n-2i}(x) - (-4)^{n+1}B_{2n+2}.$$

4.2. Dual formulas of (4) *and* (6)

We now define four kinds of enumerative polynomials over the symmetric group $\mathfrak{S}_n$:

$$a_n(x) = \sum_{k\geqslant 1} a(n,k)x^k,\ b_n(x) = \sum_{k\geqslant 0} b(n,k)x^k,$$
$$c_n(x) = \sum_{k\geqslant 1} c(n,k)x^k,\ d_n(x) = \sum_{k\geqslant 0} d(n,k)x^k,$$

where the coefficients are respectively defined by

$$a(n,k) = \#\{\pi\in\mathfrak{S}_n : \mathrm{epk} = k,\ \mathrm{ddes} = 0\},\ b(n,k) = \#\{\pi\in\mathfrak{S}_n : \mathrm{lpk} = k\},$$
$$c(n,k) = \#\{\pi\in\mathfrak{S}_n : \mathrm{epk} = k\},\ d(n,k) = \#\{\pi\in\mathfrak{S}_n : \mathrm{ipk} = k\}.$$

As pointed out by Chen and Fu [10, Theorem 10], one has $c(n,k+1) = d(n,k)$. Hence $c_n(x) = xd_n(x)$. From [11, Propositions 3.4, 4.9] and [23, Theorem 1], we see that the numbers $a(n,k), b(n,k)$ and $c(n,k)$ satisfy the following recursions:

$$\begin{cases} a(n,k) = ka(n-1,k) + (2n-4k+4)a(n-1,k-1), \\ b(n,k) = (2k+1)b(n-1,k) + (n-2k+1)b(n-1,k-1), \\ c(n,k) = 2kc(n-1,k) + (n-2k+2)c(n-1,k-1), \end{cases}$$

with $a(1,1) = c(1,1) = 1$ and $a(1,k) = c(1,k) = 0$ if $k\neq 1$, $b(1,0) = 1$ and $b(1,k) = 0$ if $k\neq 0$. Using the above recursions, it is easy to verify that $c(n,k) = 2^{n+1-2k}a(n,k)$, and so

$$c_n(x) = 2^{n+1} a_n\left(\frac{x}{4}\right).$$

For convenience, we list the first few polynomials (see [30, A101280, A008971, A008303]):

$$a_1(x) = x,\ a_2(x) = x,\ a_3(x) = x + 2x^2,\ a_4(x) = x + 8x^2;$$
$$b_1(x) = 1,\ b_2(x) = 1 + x,\ b_3(x) = 1 + 5x,\ b_4(x) = 1 + 18x + 5x^2;$$
$$c_1(x) = x,\ c_2(x) = 2x,\ c_3(x) = 4x + 2x^2,\ c_4(x) = 8x + 16x^2.$$

Theorem 7. *For $n \geqslant 1$, we have*

$$P_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a(n,k)(1+x^2)^k(2x)^{n+1-2k},\ Q_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k)(1+x^2)^k x^{n-2k}. \tag{16}$$

Proof. When $n = 1, 2$, we have $P_1(x) = 1 + x^2,\ P_2(x) = 2x(1 + x^2),\ Q_1(x) = x$ and

$$Q_2(x) = 1 + 2x^2 = x^2 + (1 + x^2).$$

So the expansions hold for $n = 1, 2$. By induction, assume that they hold for $n = m$. Then

$$\begin{aligned}
P_{m+1}(x) &= (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_m(x) \\
&= (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}\sum_k a(m,k)(1+x^2)^k(2x)^{m+1-2k} \\
&= \sum_k ka(m,k)(1+x^2)^k(2x)^{m+2-2k} + \sum_k (2m+2-4k)a(m,k)(1+x^2)^{k+1}(2x)^{m-2k},
\end{aligned}$$

which yields that the coefficient $(1 + x^2)^k(2x)^{m+2-2k}$ of $P_{m+1}(x)$ is given by

$$ka(m, k) + (2m - 4k + 6)a(m, k - 1) = a(m + 1, k),$$

as desired. Similarly,

$$\begin{aligned}
Q_{m+1}(x) &= (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}Q_m(x) + xQ_m(x) \\
&= (1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}\sum_k b(m,k)(1+x^2)^k x^{m-2k} + x\sum_k b(m,k)(1+x^2)^k x^{m-2k} \\
&= \sum_k (1+2k)b(m,k)(1+x^2)^k x^{m+1-2k} + \sum_k (m-2k)b(m,k)(1+x^2)^{k+1}x^{m-2k-1},
\end{aligned}$$

which implies that the coefficient $(1 + x^2)^k(2x)^{m+1-2k}$ of $Q_{m+1}(x)$ is given by

$$(1 + 2k)b(m, k) + (m - 2k + 2)b(m, k - 1) = b(m + 1, k).$$

This completes the proof. □

The *central factorial numbers of the second kind* $T(n, k)$ are defined in Riordan's book [29, p. 213-217] by

$$x^n = \sum_{k=0}^{n} T(n,k) x \prod_{i=1}^{k-1} \left(x + \frac{k}{2} - i \right).$$

Using central difference operator, Riordan [29, p. 214] deduced that

$$k! T(n,k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{k}{2} - j \right)^n.$$

We denote by $U(n,k) = T(2n, 2k)$ and $V(n,k) = 4^{n-k} T(2n+1, 2k+1)$ for all $n, k \geqslant 0$. These numbers satisfy the recurrence relations

$$\begin{cases} U(n,k) = U(n-1,k-1) + k^2 U(n-1,k), \\ V(n,k) = V(n-1,k-1) + (2k+1)^2 V(n-1,k), \end{cases} \tag{17}$$

with the initial conditions $U(1,1) = 1$, $U(1,k) = 0$ if $k \neq 1$, $V(0,0) = 1$ and $V(0,k) = 0$ if $k \neq 0$.

Theorem 8. *For $n \geqslant 1$, we have*

$$P_{2n-1}(x) = \sum_{j=1}^{n} (-4)^{n-j} (2j-1)! U(n,j) (1+x^2)^j, \ P_{2n}(x) = x \sum_{j=1}^{n} (-4)^{n-j} (2j)! U(n,j)(1+x^2)^j,$$

$$Q_{2n}(x) = \sum_{j=0}^{n} (-1)^{n-j} (2j)! V(n,j)(1+x^2)^j, \ Q_{2n+1}(x) = x \sum_{j=0}^{n} (-1)^{n-j} (2j+1)! V(n,j)(1+x^2)^j.$$

Proof. Note that

$$P_1(x) = 1 + x^2, \ P_2(x) = 2x(1+x^2), \ P_3(x) = -4(1+x^2) + 6(1+x^2)^2,$$
$$Q_1(x) = x, \ Q_2(x) = -1 + 2(1+x^2), \ Q_3(x) = x(-1 + 6(1+x^2)).$$

We proceed by induction. Assume that

$$P_{2m-1}(x) = \sum_{j=1}^{m} (-4)^{m-j} (2j-1)! U(m,j)(1+x^2)^j, \ Q_{2m}(x) = \sum_{j=0}^{m} (-1)^{m-j} (2j)! V(m,j)(1+x^2)^j.$$

Using (3), we arrive at

$$P_{2m}(x) = x \sum_{j=1}^{m} (-4)^{m-j} (2j)! U(m,j)(1+x^2)^j,$$
$$Q_{2m+1}(x) = x \sum_{j=0}^{m} (-1)^{m-j} (2j+1)! V(m,j)(1+x^2)^j.$$

We proceed by induction. Note that

$$\begin{aligned} P_{2m+1}(x) &= (1+x^2) \frac{\mathrm{d}}{\mathrm{d}x} P_{2m}(x) \\ &= \sum_{j=1}^{m} (-4)^{m-j} (2j)! U(m,j)(1+x^2)^{j+1} + x^2 \sum_{j=1}^{m} (-4)^{m-j} (2j)! (2j) U(m,j)(1+x^2)^j \end{aligned}$$

$$= \sum_{j=1}^{m} (-4)^{m-j}(2j)!U(m,j)(1+x^2)^{j+1} + (1+x^2-1)\sum_{j=1}^{m} (-4)^{m-j}(2j)!(2j)U(m,j)(1+x^2)^{j}$$

$$= \sum_{j=1}^{m} (-4)^{m-j}(2j)!U(m,j)(1+x^2)^{j+1} + \sum_{j=1}^{m} (-4)^{m-j}(2j)!(2j)U(m,j)(1+x^2)^{j+1} -$$

$$\sum_{j=1}^{m} (-4)^{m-j}(2j)!(2j)U(m,j)(1+x^2)^{j}$$

$$= \sum_{j=1}^{m} (-4)^{m-j}(2j+1)!U(m,j)(1+x^2)^{j+1} - \sum_{j=1}^{m} (-4)^{m-j}(2j)!(2j)U(m,j)(1+x^2)^{j}$$

$$= \sum_{j=1}^{m} (-4)^{m-j}(2j+1)!U(m,j)(1+x^2)^{j+1} + \sum_{j=1}^{m} (-4)^{m+1-j}(2j-1)!j^2U(m,j)(1+x^2)^{j}$$

Extracting the coefficient of $(-4)^{m+1-j}(2j-1)!(1+x^2)^j$ leads to

$$U(m,j-1) + j^2U(m,j) = U(m+1,j).$$

So we get

$$P_{2m+1}(x) = \sum_{j=1}^{m+1} (-4)^{m+1-j}(2j-1)!U(m+1,j)(1+x^2)^j.$$

Similarly, one can verify that

$$Q_{2m+2}(x) = \sum_{j=0}^{m+1} (-1)^{m+1-j}(2j)!V(m+1,j)(1+x^2)^j.$$

This completes the proof. □

Remarkably, combining Theorems 7 and 8, substituting $\frac{1+x^2}{4x^2} \to y$, we get the following Frobenius-type formulas for exterior peak and left peak polynomials.

Corollary 9. *For $n \geqslant 1$, we have*

$$a_{2n-1}(x) = \sum_{j=1}^{n} (2j-1)!U(n,j)x^j(1-4x)^{n-j},$$

$$a_{2n}(x) = \sum_{j=1}^{n} (2j-1)!jU(n,j)x^j(1-4x)^{n-j},$$

$$b_{2n}(x) = \sum_{j=0}^{n} (2j)!V(n,j)x^j(1-x)^{n-j},$$

$$b_{2n+1}(x) = \sum_{j=0}^{n} (2j+1)!V(n,j)x^j(1-x)^{n-j},$$

$$c_{2n-1}(x) = \sum_{j=1}^{n} (2j-1)!4^{n-j}U(n,j)x^j(1-x)^{n-j},$$

$$c_{2n}(x) = 2\sum_{j=1}^{n}(2j-1)!j4^{n-j}U(n,j)x^j(1-x)^{n-j}.$$

5. Möbius transformations of Eulerian polynomials

Let $\pm[n] = [n] \cup \{-1, -2, \ldots, -n\}$, and let B_n be the hyperoctahedral group of rank n. Elements of B_n are signed permutations of $\pm[n]$ with the property that $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$. The *type B Eulerian polynomials* are defined by

$$B_n(x) = \sum_{\sigma \in B_n} x^{\mathrm{des}_B(\sigma)},$$

where $\mathrm{des}_B(\sigma) = \#\{i \in \{0, 1, 2, \ldots, n-1\} : \ \sigma(i) > \sigma(i+1)\}$ and $\sigma(0) = 0$ (see [7] for details). It is well known that (see [16])

$$\begin{cases} \sum_{n=0}^{\infty} A_n(-1)\frac{x^n}{n!} = 1 + \tanh(x), \\ \sum_{n=0}^{\infty} B_n(-1)\frac{x^n}{n!} = \mathrm{sech}\ (2x). \end{cases} \tag{18}$$

Consider the derivative polynomials for hyperbolic tangent and secant:

$$\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\tanh\theta = \widetilde{P}_n(\tanh\theta) \quad \text{and} \quad \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\mathrm{sech}\ \theta = \mathrm{sech}\ \theta \cdot \widetilde{Q}_n(\tanh\theta).$$

It follows from $\tanh\theta = \mathrm{i}\tan(\theta/i)$ and $\mathrm{sech}\ \theta = \sec(\theta/i)$ that

$$\widetilde{P}_n(x) = \mathrm{i}^{n-1}P_n(\mathrm{i}x) \quad \text{and} \quad \widetilde{Q}_n(x) = \mathrm{i}^n Q_n(\mathrm{i}x).$$

By the chain rule, we see that

$$\begin{cases} \widetilde{P}_{n+1}(x) = (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{P}_n(x), \ \widetilde{P}_0(x) = x \\ \widetilde{Q}_{n+1}(x) = (1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{Q}_n(x) - x\widetilde{Q}_n(x), \ \widetilde{Q}_0(x) = 1. \end{cases} \tag{19}$$

Motivated by (18), we find the following result.

Theorem 10. *We have*

$$(-1)^n\widetilde{P}_n(x) = (x+1)^{n+1}A_n\left(\frac{x-1}{x+1}\right), \ (-1)^n 2^n\widetilde{Q}_n(x) = (x+1)^n B_n\left(\frac{x-1}{x+1}\right).$$

Proof. It is well known that

$$\begin{cases} A_{n+1}(x) = (nx+1)A_n(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_n(x), \\ B_{n+1}(x) = (2nx+x+1)B_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_n(x), \end{cases} \tag{20}$$

with $A_0(x) = 1$ and $B_0(x) = 1$, see [11,24]. Set

$$\widetilde{A}_n(x) = (x+1)^{n+1}A_n\left(\frac{x-1}{x+1}\right), \ \widetilde{B}_n(x) = (x+1)^n B_n\left(\frac{x-1}{x+1}\right).$$

Substituting these two expressions into (20) and simplifying, we obtain

$$\begin{cases} \widetilde{A}_{n+1}(x) = (x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{A}_n(x), \ \widetilde{A}_0(x) = x \\ \widetilde{B}_{n+1}(x) = 2(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{B}_n(x) + 2x\widetilde{B}_n(x), \ \widetilde{B}_0(x) = 1. \end{cases}$$

By (19), we obtain $\widetilde{A}_n(x) = (-1)^n \widetilde{P}_n(x)$ and $\widetilde{B}_n(x) = (-1)^n 2^n \widetilde{Q}_n(x)$, as desired. □

Acknowledgments

The third author was supported by the National Natural Science Foundation of China (Grant number 12071063) and Taishan Scholar Foundation of Shandong Province (No. tsqn202211146). The fourth author was supported by the National Science and Technology Council (Grant number: MOST 113-2115-M-017-005-MY2).

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